Gauge Invariant Pauli-Villars Regularization of Chiral Fermions

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ABSTRACT

We extend the idea of the generalized Pauli-Villars regularization of Frolov and Slavnov and analyze the general structure of the regularization scheme. The gauge anomaly-free condition emerges in a simple way in the scheme, and, under the standard prescription for the momentum assignment, the Pauli-Villars Lagrangian provides a gauge invariant regularization of chiral fermions in arbitrary anomaly-free representations. The vacuum polarization tensor is transverse, and the fermion number and the conformal anomalies have gauge invariant forms. We also point out that the real representation can be treated in a straightforward manner and the covariant regularization scheme is directly implemented.

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1. Introduction

In this paper, we extend the idea of the generalized Pauli-Villars (PV) regularization of chiral fermions proposed by Frolov and Slavnov some years ago [1]. We analyze the structure of the regularization scheme on the basis of a regularization of composite current operators, as has been performed [2] for the generalized PV regularization proposed by Narayanan and Neuberger [3]. This type of analysis provides a simple and transparent way to see the structure of the regularization of fermion one-loop diagrams.

In the past, several studies regarding this proposal were also performed: A characterization from the viewpoint of the analytic index [3], a verification of the Ward-Takahashi (WT) identities and an evaluation of the fermion number anomaly by a direct use of Feynman diagrams [4] and, more recently the generalization to arbitrary anomaly-free complex gauge representations in curved space-time [5]. We believe our formulation in this paper also provides a unified view concerning these results.

A regularization based on a diagrammatical calculation, such as the PV regularization [6], in general, preserves the Bose symmetry among external gauge vertices; thus it gives rise to the consistent [7] gauge anomaly [8]. Since the consistent anomaly is not covariant nor invariant under gauge transformation on the external gauge fields, a Bose symmetric gauge invariant regularization of chiral fermions, if possible, exists only for anomaly-free gauge representations. How this anomaly-free requirement emerges in the scheme is the main concern in the gauge invariant regularization of chiral fermions. As we will see throughout this article, the anomaly-free condition emerges in a simple way in the extension of the generalized PV scheme of Ref. [1]. This is the interesting and important property of the proposal.

The organization of this paper is as follows: In $\S 2$, we extract the essence of the Lagrangian given in Ref. [1], which was originally constructed only for the spinor representation of the SO(10) gauge group. We then present a general framework

for extension to other gauge representations. In $\S 3$, the regularized form of the composite current operators, namely the gauge current, the vector U(1) current and the axial U(1) current, and the trace part of the energy-momentum tensor are summarized.

Based on the above setting, real gauge representations are studied in detail in §4. This case allows a straightforward treatment because of the anomaly-free nature of the representation. We also point out that the resultant regularized operators are nothing but those in the covariant regularization scheme in Ref. [9].

In §5, the complex gauge representation, which is important in view of applications, is studied. This part of the paper is essentially the result given in Ref. [5] specialized to flat space-time, but we include it for the sake of completeness and for comparison with the real representation case.

In §6, as an illustration of our formulation, the fermion contribution to the vacuum polarization tensor [1,3,4,2] is calculated for arbitrary anomaly-free representations and for arbitrary regulator functions. In §7, we evaluate the fermion number anomaly [10] and the conformal anomaly [11] within our formulation. We obtain the covariant (or gauge invariant) [12,7] anomalies.

We comment briefly on the relation of the "Weyl formulation" [1] and the vector-like formulation [3] in §8. The final section is devoted to conclusions.

Throughout this article, we work in Euclidean spacetime, $ix^0 = x^4$, $A_0 = iA_4$, $i\gamma^0 = \gamma^4$ and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^4\gamma^1\gamma^2\gamma^3$, and in particular, $\gamma^{\mu\dagger} = -\gamma^{\mu}$, $\gamma_5^{\dagger} = \gamma_5$, $g_{\mu\nu} = -\delta_{\mu\nu}$ and $\varepsilon^{1234} = 1$.

2. Generalized PV Lagrangian

The PV Lagrangian due to Frolov and Slavnov [1] can be generalized as follows:

$$\mathcal{L} = \overline{\psi}i\not\!D\psi - \frac{1}{2}\overline{\psi}MC_D\overline{\psi}^T + \frac{1}{2}\psi^TC_D^{\dagger}M^{\dagger}\psi + \overline{\phi}Xi\not\!D\phi - \frac{1}{2}\overline{\phi}M'C_D\overline{\phi}^T + \frac{1}{2}\phi^TC_D^{\dagger}M'^{\dagger}\phi$$
(2.1)

In (2.1), ψ and ϕ are the fermionic and bosonic Dirac spinors, respectively, each of which possessing the gauge and an internal space indices. We shall call this internal space "generation." The first generation component of ψ , ψ_0 , is the original massless fermion to be regularized, and other components of ψ and ϕ are massive regulator fields. The number of generations may be infinite [1]. C_D is the Dirac charge conjugation matrix, and the covariant derivative \mathcal{D} is defined by

$$\not \!\!\! D \equiv \gamma^{\mu} (\partial_{\mu} - igA_{\mu}^{a} \mathcal{T}^{a} P_{R}), \quad P_{R} \equiv \frac{1 + \gamma_{5}}{2}, \tag{2.2}$$

where \mathcal{T}^a is in general a reducible representation of the gauge group.

The mass matrices M and M' in (2.1) possess gauge and the generation indices. From the statistics of the fields, these matrices must satisfy

$$M^T = M, \quad {M'}^T = -M'.$$
 (2.3)

The matrix X, which also has gauge and the generation indices, has been introduced to avoid the appearance of tachyonic bosons. Such a tachyonic state leads to an unwanted pole singularity in the regulator function f(t) (see (4.5)). Since the mass squared of the bosonic fields is given by $-X^{-1}M'X^{T^{-1}}M'^{\dagger}$, if

$$M' = -XM'X^T, (2.4)$$

then the mass squared is positive definite, $M'M'^{\dagger}$. The hermiticity of the action,

^{*} We have added free left-handed (spectator) spinors to the Lagrangian in Ref. [1]. We will only consider regularization for a *single* chiral fermion.

[†] $C_D \gamma^{\mu} C_D^{-1} = -\gamma^{\mu T}, C_D \gamma_5 C_D^{-1} = \gamma_5^T, C_D^{\dagger} = C_D^{-1} \text{ and } C_D^T = -C_D.$

on the other hand, requires

$$X^{\dagger} = X, \quad [\mathcal{T}^a, X] = 0. \tag{2.5}$$

Finally, the mass matrices should satisfy

$$\mathcal{T}^a M = -M \mathcal{T}^{aT}, \quad \mathcal{T}^a M' = -M' \mathcal{T}^{aT} \tag{2.6}$$

to be gauge invariant. Once a certain set of matrices, M, M' and X, which satisfy (2.3)–(2.6), and a suitable gauge generator \mathcal{T}^a , which is reduced to the original representation T^a on ψ_0 , are found, the gauge invariant Lagrangian (2.1) may be constructed. Our general setting (2.1) allows various extensions of Ref. [1] which will be discussed in the subsequent sections.

To reformulate the generalized PV regularization as a regularization of composite current operators, we need the formal propagators of ψ and ϕ in a fixed background gauge field. We introduce a two component notation

$$\Psi = \left(\frac{\psi}{\psi}\right), \quad \Phi = \left(\frac{\phi}{\phi}\right). \tag{2.7}$$

In terms of these variables, the Lagrangian (2.1) is written as

$$\mathcal{L} = \frac{1}{2} \Psi^T \begin{pmatrix} C_D^{\dagger} M^{\dagger} & -i \not D^T \\ i \not D & -M C_D \end{pmatrix} \Psi + \frac{1}{2} \Phi^T \begin{pmatrix} C_D^{\dagger} M'^{\dagger} & i \not D^T X^T \\ i X \not D & -M' C_D \end{pmatrix} \Phi, \tag{2.8}$$

where the transpose of the covariant derivative is defined by

$$\not \!\!\!\!D^T \equiv (-\partial_\mu - igA_\mu^a \mathcal{T}^{aT} P_R^T) \gamma^{\mu T}. \tag{2.9}$$

This definition (and analogous definitions of \mathcal{D}^{\dagger} and $\mathcal{D}^{\dagger T}$ below) is motivated by

the matrix notation in the functional space. Namely,

$$D(x,y) \equiv \gamma^{\mu}(\partial_{\mu}^{x} - igA_{\mu}^{a}(x)\mathcal{T}^{a}P_{R})\delta(x-y), \qquad (2.10)$$

and thus

$$\mathcal{D}^{T}(x,y) \equiv (\partial_{\mu}^{y} - igA_{\mu}^{a}(y)\mathcal{T}^{aT}P_{R}^{T})\gamma^{\mu T}\delta(y-x)$$

$$= (-\partial_{\mu}^{x} - igA_{\mu}^{a}(x)\mathcal{T}^{aT}P_{R}^{T})\gamma^{\mu T}\delta(x-y).$$
(2.11)

Once writing the Lagrangian in the form (2.8), it is straightforward to find the propagator in a background gauge field A^a_{μ} :

$$\left\langle T\Psi(x)\Psi^{T}(y)\right\rangle = \begin{pmatrix} -MC_{D}\frac{1}{\not D^{T}\not D^{\dagger T} + M^{\dagger}M} & i\not D^{\dagger}\frac{1}{\not D^{D}^{\dagger} + MM^{\dagger}} \\ -i\not D^{\dagger T}\frac{1}{\not D^{T}\not D^{\dagger T} + M^{\dagger}M} & C_{D}^{\dagger}M^{\dagger}\frac{1}{\not D^{D}^{\dagger} + MM^{\dagger}} \end{pmatrix} \delta(x-y),$$

$$(2.12)$$

and

$$\left\langle T\Phi(x)\Phi^{T}(y)\right\rangle = \begin{pmatrix}
-M'C_{D}\frac{1}{\cancel{D}^{T}\cancel{D}^{\dagger T}+M'^{\dagger}M'} & i\cancel{D}^{\dagger}X^{-1}\frac{1}{\cancel{D}\cancel{D}^{\dagger}+M'M'^{\dagger}} \\
iX^{-1}^{T}\cancel{D}^{\dagger T}\frac{1}{\cancel{D}^{T}\cancel{D}^{\dagger T}+M'^{\dagger}M'} & C_{D}M'^{\dagger}\frac{1}{\cancel{D}\cancel{D}^{\dagger}+M'M'^{\dagger}}
\end{pmatrix} \delta(x-y). \tag{2.13}$$

In the above expressions, we have introduced

$$\not\!\!D^{\dagger} \equiv (\partial_{\mu} - igA_{\mu}^{a} \mathcal{T}^{a} P_{R}) \gamma^{\mu}
= \gamma^{\mu} (\partial_{\mu} - igA_{\mu}^{a} \mathcal{T}^{a} P_{L}) \neq \not\!\!D,$$
(2.14)

where $P_L \equiv (1 - \gamma_5)/2$, and

$$\mathcal{D}^{\dagger T} \equiv -\gamma^{\mu T} (\partial_{\mu} + igA^{a}_{\mu} \mathcal{T}^{aT} P^{T}_{R}). \tag{2.15}$$

It is interesting to note that the hermitian conjugate of the covariant derivative automatically emerges in the inverse operator.

[‡] In deriving these formulas, it is necessary to use relations such as $\not \!\! D^\dagger C_D^\dagger M = C_D^\dagger M \not \!\! D^T$, $\not \!\! D^\dagger C_D^\dagger M^\dagger = C_D^\dagger M^\dagger \not \!\! D$, etc. These follow from the gauge invariance of the mass term, (2.6).

3. Composite current operators in PV regularization

The central quantity in our analysis is the gauge current, whose classical form is defined by a functional derivative of the action (2.1) with respect to the gauge field:

$$\exists \frac{1}{2} \Psi^T \begin{pmatrix} 0 & -\mathcal{T}^{aT} P_R^T \gamma^{\mu T} \\ \gamma^{\mu} \mathcal{T}^a P_R & 0 \end{pmatrix} \Psi + \frac{1}{2} \Phi^T \begin{pmatrix} 0 & \mathcal{T}^{aT} P_R^T \gamma^{\mu T} X^T \\ X \gamma^{\mu} \mathcal{T}^a P_R & 0 \end{pmatrix} \Phi.$$
(3.1)

Therefore in the PV regularization by Frolov and Slavnov, the regularized gauge current is defined by

$$\langle J^{\mu a}(x) \rangle_{PV} = \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[(-1) \begin{pmatrix} 0 & -\mathcal{T}^{aT} P_R^T \gamma^{\mu T} \\ \gamma^{\mu} \mathcal{T}^a P_R & 0 \end{pmatrix} \left\langle \mathcal{T} \Psi(x) \Psi^T(y) \right\rangle \right. \\ \left. + \begin{pmatrix} 0 & \mathcal{T}^{aT} P_R^T \gamma^{\mu T} X^T \\ X \gamma^{\mu} \mathcal{T}^a P_R & 0 \end{pmatrix} \left\langle \mathcal{T} \Phi(x) \Phi^T(y) \right\rangle \right] \\ = \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \mathcal{T}^a P_R i \not D^{\dagger} \frac{-1}{\not D \not D^{\dagger} + M M^{\dagger}} \delta(x - y) \right. \\ \left. + \gamma^{\mu} \mathcal{T}^a P_R i \not D^{\dagger} \frac{1}{\not D \not D^{\dagger} + M' M'^{\dagger}} \delta(x - y) \right],$$

$$(3.2)$$

where the trace is taken over the generation, gauge and Dirac indices. We note that this definition is in accord with the standard Feynman diagrammatical calculation: Further derivatives of (3.2) with respect to the background gauge field gives a multipoint one loop vertex function. We will later illustrate such a calculation of the vacuum polarization tensor. Equation (3.2) therefore summarizes the structure of the regularization scheme in a neat way.

Strictly speaking, a PV Lagrangian such as (2.1) alone cannot definitely specify the regularization scheme. As is well-known, one must supplement the following prescriptions [6] to the Lagrangian: 1) The integrand of the momentum integration must be summed over all the generations prior to the momentum integration. 2) The momentum assignment for all the fields (the original fermion and the regulators) should be taken the same. It is thus important in (3.2) that the trace over the generation index be taken before the equal point limit $y \to x$, according to prescription 1). Equation (3.2) is as it stands a formal implemention of prescription 2): The momentum assignment is common for all the generations. However, one should always be careful in the uniform momentum assignment in actual calculations such as (6.4). Those two underlying prescriptions in the PV regularization are understood throughout this paper. With this caution in mind, we use the term "Lagrangian level regularization."

Another important composite current operator in the chiral gauge theory is the fermion number current. We define it as the Noether current associated with a global U(1) rotation [4,2]:

$$\psi(x) \to e^{i\alpha} \psi(x), \quad \overline{\psi}(x) \to \overline{\psi}(x) e^{-i\alpha},$$

$$\phi(x) \to e^{i\alpha} \phi(x), \quad \overline{\phi}(x) \to \overline{\phi}(x) e^{-i\alpha},$$
(3.3)

or, in terms of the two component notation,

$$\Psi(x) \to \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \Psi(x), \quad \Phi(x) \to \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \Phi(x).$$
(3.4)

By localizing the infinitesimal parameter α , the Noether current is defined by

$$\mathcal{L} \to \mathcal{L} - (\partial_{\mu}\alpha)J^{\mu}(x) - \alpha B(x),$$
 (3.5)

where

$$J^{\mu}(x) \equiv \frac{1}{2} \Psi^T \begin{pmatrix} 0 & -\gamma^{\mu T} \\ \gamma^{\mu} & 0 \end{pmatrix} \Psi + \frac{1}{2} \Phi^T \begin{pmatrix} 0 & \gamma^{\mu T} X^T \\ X \gamma^{\mu} & 0 \end{pmatrix} \Phi, \tag{3.6}$$

and the explicit breaking part B(x) is given by

$$B(x) \equiv \frac{1}{2} \Psi^T \begin{pmatrix} -2iC_D^{\dagger} M^{\dagger} & 0\\ 0 & -2iMC_D \end{pmatrix} \Psi + \frac{1}{2} \Phi^T \begin{pmatrix} -2iC_D^{\dagger} {M'}^{\dagger} & 0\\ 0 & -2i{M'}C_D \end{pmatrix} \Phi.$$
(3.7)

By defining the composite current operator by the propagators, we have from (3.6),

$$\langle J^{\mu}(x)\rangle_{PV} = \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[(-1) \begin{pmatrix} 0 & -\gamma^{\mu T} \\ \gamma^{\mu} & 0 \end{pmatrix} \left\langle T\Psi(x)\Psi^{T}(y) \right\rangle \right. \\ \left. + \begin{pmatrix} 0 & \gamma^{\mu T} X^{T} \\ X\gamma^{\mu} & 0 \end{pmatrix} \left\langle T\Phi(x)\Phi^{T}(y) \right\rangle \right]$$

$$= \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} i \not D^{\dagger} \frac{-1}{\not D \not D^{\dagger} + MM^{\dagger}} \delta(x - y) \right. \\ \left. + \gamma^{\mu} i \not D^{\dagger} \frac{1}{\not D \not D^{\dagger} + M'M'^{\dagger}} \delta(x - y) \right].$$

$$\left. + \gamma^{\mu} i \not D^{\dagger} \frac{1}{\not D \not D^{\dagger} + M'M'^{\dagger}} \delta(x - y) \right].$$

If the composite current operator is well regularized, we may derive the corresponding WT identity,

$$\partial_{\mu} \langle J^{\mu}(x) \rangle_{PV} = \langle B(x) \rangle_{PV},$$
 (3.9)

as a result of the naive equation of motion. Under a Lagrangian level regularization, a possible anomaly associated with a certain global symmetry should arise as an *explicit* symmetry breaking term in the Lagrangian. Thus the vacuum expectation value of B(x), the right-hand side of (3.9), gives rise to the fermion number anomaly. We will later verify that this is, in fact, the case. This situation is analogous to an evaluation of the gauge anomaly in chiral gauge theories by the *conventional* PV regularization, where the *gauge* symmetry is explicitly broken by the PV mass term.

We may equally well use another definition of the regularized fermion number current. It is defined by the Noether current associated with a global $axial\ U(1)$

rotation [4,2]:

$$\psi(x) \to e^{i\alpha\gamma_5}\psi(x), \quad \overline{\psi}(x) \to \overline{\psi}(x)e^{i\alpha\gamma_5},$$

$$\phi(x) \to e^{i\alpha\gamma_5}\phi(x), \quad \overline{\phi}(x) \to \overline{\phi}(x)e^{i\alpha\gamma_5}.$$
(3.10)

By the same procedure as above, we find the associated Noether current

$$\langle J_5^{\mu}(x) \rangle_{PV} = \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \gamma_5 i \not \!\! D^{\dagger} \frac{-1}{\not \!\! D \not \!\! D^{\dagger} + M M^{\dagger}} \delta(x - y) + \gamma^{\mu} \gamma_5 i \not \!\! D^{\dagger} \frac{1}{\not \!\! D \not \!\! D^{\dagger} + M' M'^{\dagger}} \delta(x - y) \right].$$

$$(3.11)$$

In what follows, we find that it is always possible to choose M and M', such that composite operators in (3.8) and (3.11) are regularized, if the gauge representation is free of the gauge anomaly. In this case, we can see that the currents (3.8) and (3.11) are the same object: We first note from (2.2) and (2.14) that

$$\frac{1}{D\!\!\!/D^{\dagger} + MM^{\dagger}} = P_L \frac{1}{D\!\!\!/D^{\dagger} + MM^{\dagger}} + P_R \frac{1}{\partial^2 + MM^{\dagger}}, \tag{3.12}$$

holds, as does an analogous relation for the bosonic part. Putting these into (3.8) and (3.11) and noting that there exists no constant vector independent of A_{μ} , we see that only the first term of (3.12) survives; consequently (3.8) and (3.11) are the same operator (note $\gamma_5 P_R = P_R$). Of course this is an expected result because only the right-handed fields are coupled to the background gauge field. After observing the equivalence of (3.8) and (3.11), we use (3.8) as the fermion number current in what follows.

Another interesting operator is the trace part of the energy-momentum tensor $T^{\mu}_{\mu}(x)$, which is defined in the original theory by

$$\overline{\psi}_0 i \not\!\!D \psi_0 \to \overline{\psi}_0 i \not\!\!D \psi_0 - \alpha(x) T^{\mu}_{\mu}(x), \quad T^{\mu}_{\mu}(x) \equiv \overline{\psi}_0 \frac{i}{2} \not\!\!D \psi_0, \tag{3.13}$$

[★] This definition requires some explanation: If one simply uses the standard definition of the energy-momentum tensor of the spinor field, −3 times our result will be obtained. Our definition, following Ref. [13], is specified by the *general coordinate* invariance in the background gravitational field (see Refs. [13,2] for more details).

where the variation of the field is

$$\psi_0(x) \to e^{-\alpha(x)/2} \psi_0(x), \quad \bar{\psi}_0(x) \to \bar{\psi}_0(x) e^{-\alpha(x)/2}.$$
 (3.14)

By generalizing the rescaling of the field (3.14) to all the regulator fields, the regularized version of the trace part of the energy-momentum tensor is defined by

$$\begin{split} \left\langle T^{\mu}_{\mu}(x) \right\rangle_{PV} &\equiv \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[(-1) \begin{pmatrix} 0 & -\frac{i}{2} \not D_{x}^{T} \\ \frac{i}{2} \not D_{x} & 0 \end{pmatrix} \left\langle T \Psi(x) \Psi^{T}(y) \right\rangle \right. \\ & \left. + (-1) \left\langle T \Psi(x) \Psi^{T}(y) \right\rangle \begin{pmatrix} 0 & \frac{i}{2} \not D_{y}^{T} \\ -\frac{i}{2} \not D_{y} & 0 \end{pmatrix} \right. \\ & \left. + \begin{pmatrix} 0 & \frac{i}{2} \not D_{x}^{T} X^{T} \\ \frac{i}{2} X \not D_{x} & 0 \end{pmatrix} \left\langle T \Phi(x) \Phi^{T}(y) \right\rangle \\ & \left. + \left\langle T \Phi(x) \Phi^{T}(y) \right\rangle \begin{pmatrix} 0 & -\frac{i}{2} \not D_{y}^{T} X^{T} \\ -\frac{i}{2} X \not D_{y} & 0 \end{pmatrix} \right]. \end{split}$$

$$(3.15)$$

In (3.13) and (3.15), $\stackrel{\leftrightarrow}{\not}{\mathbb{D}} = \not{\mathbb{D}} - \stackrel{\leftarrow}{\not}{\mathbb{D}}, \stackrel{\leftrightarrow}{\not}{\mathbb{D}}^T = \not{\mathbb{D}}^T - \stackrel{\leftarrow}{\not}{\mathbb{D}}^T$ and

$$\stackrel{\leftarrow}{\not\!\!\!D} \equiv \gamma^{\mu} (\stackrel{\leftarrow}{\partial_{\mu}} + ig A^{a}_{\mu} \mathcal{T}^{a} P_{R}), \quad \stackrel{\leftarrow}{\not\!\!\!D}^{T} \equiv (-\stackrel{\leftarrow}{\partial_{\mu}} + ig A^{a}_{\mu} \mathcal{T}^{aT} P_{R}^{T}) \gamma^{\mu T}.$$
(3.16)

$$\frac{1}{\not \!\!\!D \not \!\!\!D^{\dagger} + MM^{\dagger}} \not \!\!\!D = \not \!\!\!D \frac{1}{\not \!\!\!D^{\dagger} \not \!\!\!\!D + MM^{\dagger}}, \tag{3.17}$$

we finally have

$$\begin{split} \left\langle T^{\mu}_{\mu}(x) \right\rangle_{PV} &= \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\not{\!\!D} \not{\!\!D}^{\dagger} \frac{1}{\not{\!\!D} \not{\!\!D}^{\dagger} + MM^{\dagger}} \delta(x-y) + \not{\!\!D} \not{\!\!D}^{\dagger} \frac{-1}{\not{\!\!D} \not{\!\!D}^{\dagger} + M'M'^{\dagger}} \delta(x-y) \right] \\ &+ \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\not{\!\!D}^{\dagger} \not{\!\!D} \frac{1}{\not{\!\!D}^{\dagger} \not{\!\!D} + MM^{\dagger}} \delta(x-y) + \not{\!\!D}^{\dagger} \not{\!\!D} \frac{-1}{\not{\!\!D}^{\dagger} \not{\!\!D} + M'M'^{\dagger}} \delta(x-y) \right]. \end{split} \tag{3.18}$$

The composite operators, the gauge current (3.2), the fermion number cur-

rent (3.8), and the trace part of the energy-momentum tensor (3.18) will be analyzed in detail in the following discussion.

4. Real representations

As was noted in the Introduction, the gauge invariant regularization of a chiral fermion is possible only for anomaly-free gauge representations. The situation is simple for real representations, because the anomaly-free condition is always fulfilled by the presence of a matrix U which transforms the original representation to the adjoint representation. As we will see below, the generalized PV Lagrangian (2.1) can utilize this fact, and this is an advantage of the present framework.

For any real representation T^a , there exists a unitary matrix U such that

$$-T^{aT} = -T^{a*} = UT^aU^{\dagger}. \tag{4.1}$$

For a real-positive representation, U is symmetric, and for a pseudo-real representation, U is anti-symmetric. For both cases, we can make the choice:

$$T^a \equiv T^a \otimes 1, \quad M = U^{\dagger} \otimes m, \quad M' = U^{\dagger} \otimes m', \quad X = 1 \otimes x,$$
 (4.2)

where the first index acts on the gauge and the second acts on the generation index. It turns out that the nature of m and m' are quite different depending on whether the representation is real-positive or pseudo-real. We thus treat them separately.

When the chiral fermion belongs to a real-positive representation of the gauge group, we can take, for example,

$$m = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \Lambda, \quad m' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Lambda, \quad x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.3}$$

where Λ is the cutoff parameter. It is readily verified that these matrices satisfy (2.3)–(2.6). Since $MM^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \Lambda^2$ and $M'M'^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Lambda^2$, the regularized gauge

current (3.2) is given by

$$\langle J^{\mu a}(x)\rangle_{PV} = \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} T^{a} P_{R} \mathcal{D}^{\dagger} \frac{1}{i \mathcal{D} \mathcal{D}^{\dagger}} f(\mathcal{D} \mathcal{D}^{\dagger} / \Lambda^{2}) \delta(x - y) \right], \tag{4.4}$$

where we have defined the regulator function

$$f(t) \equiv \frac{2}{(t+1)(t+2)},\tag{4.5}$$

which vanishes rapidly as $t \to \infty$ and satisfies

$$f(0) = 1, \quad \lim_{t \to 0} t f'(t) = \lim_{t \to 0} t^2 f''(t) = \lim_{t \to 0} t^3 f^{(3)}(t) = 0,$$

$$\lim_{t \to \infty} t f(t) = \lim_{t \to \infty} t^2 f'(t) = \lim_{t \to \infty} t^2 f''(t) = \lim_{t \to \infty} t^3 f^{(3)}(t) = 0.$$
(4.6)

Due to the rapid damping property in the second line, the gauge current (4.4) is well regularized, and the limit $y \to x$ can safely be taken.

Let us next consider the pseudo-real case, for which the matrix U is anti-symmetric. In (4.2) we may choose

$$m = \begin{pmatrix} 0 & & & & & \\ & 0 & 2 & & & \\ & -2 & 0 & & & \\ & & & 0 & 4 & \\ & & & -4 & 0 & \\ & & & & \ddots \end{pmatrix} \Lambda, \quad m' = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 3 & \\ & & 3 & 0 & \\ & & & \ddots \end{pmatrix} \Lambda, \quad (4.7)$$

and

$$x = diag(1, -1, 1, -1, \cdots). \tag{4.8}$$

These matrices again satisfy (2.3)–(2.6). Note that in this case we have introduced

an infinite number of regulator fields. The mass squared is given by

$$MM^{\dagger} = 1 \otimes \begin{pmatrix} 0^{2} & & & \\ & 2^{2} & & \\ & & 2^{2} & \\ & & & \ddots \end{pmatrix} \Lambda^{2}, \quad M'M'^{\dagger} = 1 \otimes \begin{pmatrix} 1^{2} & & & \\ & 1^{2} & & \\ & & 3^{2} & & \\ & & & 3^{2} & \\ & & & & \ddots \end{pmatrix} \Lambda^{2}.$$

$$(4.9)$$

With the above choice of mass matrices, the gauge current operator (3.2) becomes

$$\begin{split} \langle J^{\mu a}(x) \rangle_{PV} &= \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} T^{a} P_{R} \not \!\!\!D^{\dagger} \frac{1}{i \not \!\!\!D \not \!\!\!D^{\dagger}} \sum_{n = -\infty}^{\infty} \frac{(-1)^{n} \not \!\!\!D \not \!\!\!D^{\dagger}}{\not \!\!\!D \not \!\!\!D^{\dagger} + n^{2} \Lambda^{2}} \delta(x - y) \right] \\ &= \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} T^{a} P_{R} \not \!\!\!D^{\dagger} \frac{1}{i \not \!\!\!D \not \!\!\!D^{\dagger}} f(\not \!\!\!D \not \!\!\!D^{\dagger} / \Lambda^{2}) \delta(x - y) \right], \end{split} \tag{4.10}$$

where the regulator function f(t) is defined by [1]

$$f(t) \equiv \sum_{n=-\infty}^{\infty} \frac{(-1)^n t}{t + n^2} = \frac{\pi \sqrt{t}}{\sinh(\pi \sqrt{t})},\tag{4.11}$$

which again has the desired properties (4.6).*

In (4.4) and (4.10), we see that all the divergences are made finite gauge invariantly. In fact these expressions are nothing but those of the covariant regularization scheme [9], which is formulated as a gauge invariant damping factor $f(\not D \not D^{\dagger}/\Lambda^2)$

$$m = \begin{pmatrix} 0 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 4 & & \\ & & & 4 & & \\ & & & \ddots \end{pmatrix} \Lambda, \quad m' = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 3 & \\ & & -3 & 0 & \\ & & & \ddots \end{pmatrix} \Lambda,$$

and x in (4.8) may be chosen.

^{*} Although it is not necessarily required, if one prefers the regulator function (4.11) in the real-positive case,

insertion in the original fermion propagator. The covariant regularization is known to give the covariant anomaly [12,7]:

$$D_{\mu} \langle J^{\mu a}(x) \rangle_{PV} \stackrel{\Lambda \to \infty}{=} \frac{ig^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} \operatorname{tr} \left(T^a F_{\mu\nu} F_{\rho\sigma} \right), \tag{4.12}$$

where the field strength is defined by $F_{\mu\nu} \equiv \left(\partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + gf^{abc}A^{b}_{\mu}A^{c}_{\nu}\right)T^{a}$ and the right-hand side vanishes due to the anomaly-free condition $\operatorname{tr}(T^{a}\{T^{b}, T^{c}\}) = 0^{\ddagger}$. Starting from (4.4) or (4.10), we can directly evaluate the gauge anomaly (4.12) and the calculation is almost identical to the passage from (7.1) to (7.6) [15]. The regularization scheme due to Frolov and Slavnov therefore gives a Lagrangian level implementation of the covariant regularization scheme in Ref. [9]. This aspect of the generalized PV regularization is studied in detail in Ref. [15].

Going back to consideration of the pseudo-real case, we have chosen the mass matrices (4.7) simply because the explicit summation over the generation index can be performed as (4.11). We may make other choice of m and m' in (4.2), and it may be thought that a more clever choice could reduce the number of regulator fields to finite value. We show below, however, that an infinite number of regulator fields are always needed, at least when relying on the construction (4.2).

Let us first assume the number of the generation is finite, and m, m' and x are finite dimensional matrices. From (2.3), $m^T = -m$ (note U is anti-symmetric). Therefore if m is an even dimensional matrix, it may be put into a block diagonal

[†] The vacuum overlap approach [14] in the lattice chiral gauge theory, which is closely related to the generalized PV regularization [3], is known to give the consistent anomaly.

[‡] Hence it trivially satisfies the Wess-Zumino consistency condition. This is consistent with the fact that we are treating a Lagrangian level regularization which respects the Bose symmetry among gauge vertices.

[§] If all the fields belong to the same irreducible representation, (2.5) and Schur's lemma imply the structure $X = 1 \otimes x$.

form by an orthogonal transformation of ψ :

$$m = \begin{pmatrix} m_1 \otimes \varepsilon & & & \\ & m_2 \otimes \varepsilon & & \\ & & \ddots & \\ & & & m_k \otimes \varepsilon \end{pmatrix}, \quad \varepsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
(4.13)

and

$$MM^{\dagger} = 1 \otimes \begin{pmatrix} m_1^2 \otimes I & & & \\ & m_2^2 \otimes I & & \\ & & \ddots & \\ & & & m_k^2 \otimes I \end{pmatrix}, \quad I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.14}$$

The number of massless fermion fields, if such fields exist, is always even. Since we are constructing a regularization for a *single* massless fermion, the matrix m should be odd dimensional. On the other hand, from (2.5), x is hermitian, and by a unitary transformation and a rescaling of ϕ , it may be put into the form

$$x = \operatorname{diag}(\underbrace{1, 1, \cdots, 1}_{k}, \underbrace{-1, -1, \cdots, -1}_{l}).$$
 (4.15)

Then (2.4) implies $m'_{ij} = -x_i x_j m'_{ij}$, and m' has the structure

$$m' = \begin{pmatrix} 0 & Y \\ Y^T & 0 \end{pmatrix} \Lambda, \tag{4.16}$$

where Y is a $k \times l$ matrix. If k < l, dim ker $Y \ge l - k$, and if k > l, dim ker $Y^T \ge k - l$. For both of these cases, there exists at least one linear combination of ϕ_i which remains massless. This is an unwanted massless bosonic field, and we should take k = l. Therefore x and m' must be even dimensional.

We now have an odd number of fermions and an even number of bosons in the same gauge representation. However, the PV condition for the ultraviolet divergence reduction [6] always requires the same numbers of fermionic and bosonic degrees of freedom. From the above argument, this is impossible when the number of the generation is finite. This shows that, at least within the construction (4.2), an infinite number of regulator fields are always needed. As we have observed, they in fact regularize the theory.

By the same procedure as for the gauge current, the regularized U(1) global current (3.8) becomes

$$\langle J^{\mu}(x)\rangle_{PV} = \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \not \!\! D^{\dagger} \frac{1}{i \not \!\! D \not \!\! D^{\dagger}} f(\not \!\! D \not \!\! D^{\dagger} / \Lambda^2) \delta(x - y) \right], \tag{4.17}$$

with an appropriate regulator function f(t). Similarly, the trace part of the energy-momentum tensor (3.18) is given by

$$\left\langle T^{\mu}_{\mu}(x)\right\rangle_{PV} = \frac{1}{2} \lim_{y \to x} \operatorname{tr}\left[f(\cancel{D}\cancel{D}^{\dagger}/\Lambda^2)\delta(x-y) + f(\cancel{D}^{\dagger}\cancel{D}/\Lambda^2)\delta(x-y) \right]. \tag{4.18}$$

Therefore, for real representations, the generalized PV Lagrangian provides a complete gauge invariant regularization of the gauge current (4.4) or (4.10), as well as the fermion number current (4.17) and the trace part of the energy-momentum tensor (4.18).

5. Complex representations

The generalized PV regularization [1] was originally formulated for the irreducible spinor representation of SO(10), i.e., an anomaly-free complex representation, which is important for an application to the standard model. Quite recently [5] the construction has been successfully generalized for arbitrary anomaly-free complex representations. We include these results in this section (specializing them to flat space-time) and compare the situation with that of the real representation in the previous section.

For a generalization of the PV Lagrangian to arbitrary complex representations, it is crucial to introduce a doubled representation [5]:

$$\mathcal{T}^a = \begin{pmatrix} T^a & 0\\ 0 & -T^{a*} \end{pmatrix} \otimes 1. \tag{5.1}$$

With this doubling of the gauge representation, the following choice of matrices in (2.1) satisfies (2.3)–(2.6):

$$M = \sigma^1 \otimes m, \quad M' = i\sigma^2 \otimes m', \quad X = \sigma^3 \otimes 1,$$
 (5.2)

where σ^i is the Pauli matrix. The relation to the original SO(10) model [1] is

$$\sigma^1 \to \Gamma_{11}C, \quad i\sigma^2 \to C, \quad \sigma^3 \to \Gamma_{11}.$$
 (5.3)

The matrices m and m' are chosen as

$$m = \begin{pmatrix} 0 & & & \\ & 2 & & \\ & & 4 & \\ & & & \ddots \end{pmatrix} \Lambda, \quad m' = \begin{pmatrix} 1 & & & \\ & 3 & & \\ & & 5 & \\ & & & \ddots \end{pmatrix} \Lambda. \tag{5.4}$$

Although the regulator fields must belong to the doubled representation (5.1) to

^{*} The gamma matrix for the spinor representation satisfies the Clifford algebra $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$ and is hermitian. The gauge generator is defined by $\mathcal{T}^a = i[\Gamma_i, \Gamma_j]/2$ and the irreducible representation is projected by $(1 + \Gamma_{11})/2$. The "chiral" matrix Γ_{11} is defined by $\Gamma_{11} = -i\Gamma_1\Gamma_2\cdots\Gamma_{10}$ and satisfies $\{\Gamma_{11}, \Gamma_i\} = 0$, $\Gamma_{11}^{\dagger} = \Gamma_{11}$ and $\Gamma_{11}^2 = 1$. The "charge conjugation" matrix C has the properties $C\Gamma_iC^{-1} = -\Gamma_i^T$, $C\Gamma_{11}C^{-1} = -\Gamma_{11}^T$, $C^{\dagger} = C^{-1}$ and $C^T = -C$.

have a non-vanishing mass, the original massless fermion ψ_0 must be projected by $(1+\sigma^3)/2$ to have the original complex representation T^a , rather than the doubled representation T^a .

Let us now consider the regularized composite operators. For example, the regularized gauge current (3.2) is given by

$$\langle J^{\mu a}(x) \rangle_{PV}$$

$$= \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \mathcal{T}^{a} P_{R} \not \!\!{D}^{\dagger} \frac{1}{i \not \!\!{D}} \not \!\!{D}^{\dagger} \sum_{n = -\infty}^{\infty} \frac{(-1)^{n} \not \!\!{D}} \not \!\!{D}^{\dagger} + n^{2} \Lambda^{2}}{\not \!\!{D}} \delta(x - y) \right]$$

$$+ \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \sigma^{3} \mathcal{T}^{a} P_{R} \not \!\!{D}^{\dagger} \frac{1}{i \not \!\!{D}} \not \!\!{D}^{\dagger} \delta(x - y) \right]$$

$$= \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \mathcal{T}^{a} P_{R} \not \!\!{D}^{\dagger} \frac{1}{i \not \!\!{D}} \not \!\!{D}^{\dagger} / \Lambda^{2}) \delta(x - y) \right]$$

$$+ \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \sigma^{3} \mathcal{T}^{a} P_{R} \frac{1}{i \not \!\!{D}} \delta(x - y) \right],$$

$$(5.5)$$

where the regulator function is given by (4.11) and we have used the fact that the first generation fermion is projected by $(1 + \sigma^3)/2$.

Similarly, the fermion number current (3.8) and the trace part of the energy-momentum tensor (3.18) become

$$\langle J^{\mu}(x)\rangle_{PV} = \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \not \!\!\!D^{\dagger} \frac{1}{i \not \!\!\!D} \not \!\!\!D^{\dagger} / \Lambda^{2}) \delta(x - y) \right] + \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \sigma^{3} \frac{1}{i \not \!\!\!D} \delta(x - y) \right]$$

$$(5.6)$$

and

$$\left\langle T^{\mu}_{\mu}(x)\right\rangle_{PV} = \frac{1}{2} \cdot \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[f(\not \!\!\!D \not \!\!\!D^{\dagger}/\Lambda^2) \delta(x-y) + f(\not \!\!\!D^{\dagger} \not \!\!\!D/\Lambda^2) \delta(x-y) \right]. \quad (5.7)$$

In deriving (5.5)–(5.7), we have taken the trace over the generation index. In the final line of (5.5) and of (5.6), the first term is completely regularized. Since the regulator function f(t) damps rapidly enough as $t \to \infty$, the limit $y \to x$ can safely

be taken to give the finite result. It should also be noted that an infinite number of regulator fields are always needed to balance the fermionic and the bosonic degrees of freedom [1]. On the other hand, the last terms of (5.5) and (5.6) do not yet have a regulator function, and the expression is ill-defined in general. Compare these with (4.10) and (4.17) for the real representations. The origin of the complication in the complex representation is that it is in general not anomaly-free, and it is impossible to distinguish the anomaly-free representations from the anomalous ones at the level of Lagrangian construction. Therefore we must supplement the regularization scheme with the anomaly-free condition.

In fact, as first pointed out in Ref. [1] (see also Refs. [3,4]) and generalized in Ref. [5], these unregularized terms become finite if and only if $\operatorname{tr} T^a = 0$ and $\operatorname{tr} T^a \{T^b, T^c\} = 0$, i.e., if free of the gauge (and Lorentz) anomaly. To see this, we note the perturbative expansion

$$\frac{1}{i\mathcal{D}} = \frac{1}{i\partial} + \frac{1}{i\partial}(-g\mathcal{A}P_R)\frac{1}{i\partial} + \frac{1}{i\partial}(-g\mathcal{A}P_R)\frac{1}{i\partial}(-g\mathcal{A}P_R)\frac{1}{i\partial} + \cdots.$$
 (5.8)

Using $\delta(x-y) = \int d^4k \, e^{-ik(x-y)}/(2\pi)^4$ and the momentum representation of $A^a_\mu(x)$, we see that the last terms in (5.5) and (5.6) generate Feynman integrals with an insertion of σ^3 . The momentum integration of the one, two, three and four-point functions is power-counting divergent, and higher point functions are convergent. Therefore if

$$\operatorname{tr}(\sigma^3 \mathcal{T}^{a_1} \cdots \mathcal{T}^{a_n}) = \operatorname{tr}(T^{a_1} \cdots T^{a_n}) + (-1)^{n+1} \operatorname{tr}(T^{a_n} \cdots T^{a_1}) = 0, \text{ for } n \le 4,$$
(5.9)

the integrand of the power-counting divergent expression vanishes. As is easily verified [4,5], this condition is equivalent to $\operatorname{tr} T^a = 0$ and $\operatorname{tr}(T^a\{T^b, T^c\}) = 0$. In the framework of Ref. [1], the anomaly-free complex representation is distinguished from anomalous ones in this way, and the generalized PV regularization works only for the anomaly-free case, as should be the case.*

 $[\]star$ When the construction of this section is applied to *real* representations [5], Eq. (5.9) holds for *all* n due to the presence of the matrix U.

6. Vacuum polarization tensor

As an illustration, we compute the fermion contribution to the vacuum polarization tensor [1,4,2] in our formulation, (4.10) or (5.5). For simplicity, we first consider the case of the real representation (4.10) and comment later on the anomaly-free complex representation (5.5).

The vacuum polarization tensor is defined by the first functional derivative of the gauge current (4.10) with respect to the background gauge field:

$$\frac{\delta \langle J^{\mu a}(x) \rangle_{PV}}{\delta (gA_{\nu}^{b}(z))} \bigg|_{A=0} \equiv \int \frac{d^{4}q}{(2\pi)^{4}} e^{-iq(x-z)} \Pi^{\mu\nu ab}(q). \tag{6.1}$$

We assume the following form of the regulator function:

$$\widetilde{f}(t) = \sum_{n} \frac{c_n t}{t + m_n^2 / \Lambda^2}.$$
(6.2)

The above examples, f(t) in (4.5), f(t) in (4.11), and the generalized PV in general, are certainly contained in this class of functions. We also assume that c_n and m_n are chosen so as to satisfy (4.6). From the definition of the covariant derivative, (2.2) and (2.14), we have

$$\frac{\delta \langle J^{\mu a}(x) \rangle_{PV}}{\delta(gA_{\nu}^{b}(z))} \Big|_{A=0}$$

$$= -\operatorname{tr}(T^{a}T^{b}) \lim_{y \to x} \operatorname{tr} \left(P_{L}\gamma^{\mu} \left\{ \delta(x-z)\gamma^{\nu} \sum_{n} \frac{c_{n}}{\Box + m_{n}^{2}} + i \partial \sum_{n} \frac{c_{n}}{\Box + m_{n}^{2}} \left[i \partial \delta(x-z)\gamma^{\nu} + \delta(x-z)\gamma^{\nu} i \partial \right] \frac{1}{\Box + m_{n}^{2}} \right\} \delta(x-y) \right).$$
(6.3)

All the derivatives in (6.3) act on everything to their right. We then use the momentum representation of the delta functions, $\delta(x-z) = \int d^4q \, e^{-iq(x-z)}/(2\pi)^4$ etc.

The vacuum polarization tensor (6.1) is then given by

$$\Pi^{\mu\nu ab}(q) = -\operatorname{tr}(T^{a}T^{b}) \int \frac{d^{4}k}{(2\pi)^{4}} \left\{ \operatorname{tr}(P_{L}\gamma^{\mu}\gamma^{\nu}) \sum_{n} \frac{c_{n}}{-(k+q)^{2} + m_{n}^{2}} \frac{m_{n}^{2}}{-k^{2} + m_{n}^{2}} + \operatorname{tr}[P_{L}\gamma^{\mu}(\not k + \not q)\gamma^{\nu}\not k] \sum_{n} \frac{c_{n}}{-(k+q)^{2} + m_{n}^{2}} \frac{1}{-k^{2} + m_{n}^{2}} \right\}.$$
(6.4)

The subsequent steps are standard: We introduce the Feynman parameter to combine the denominators, shift the integration momentum, * and take the trace of the gamma matrices. Noting the definition of $\widetilde{f}(t)$ (6.2), we have

$$\Pi^{\mu\nu ab}(q) = -\operatorname{tr}(T^{a}T^{b}) \int_{0}^{1} dx \int \frac{d^{4}k}{(2\pi)^{4}} \\
\times \left\{ g^{\mu\nu} \left[2\sum_{n} \frac{c_{n}}{-k^{2} - q^{2}x(1-x) + m_{n}^{2}} + \sum_{n} \frac{c_{n}k^{2}}{[-k^{2} - q^{2}x(1-x) + m_{n}^{2}]^{2}} \right] \\
- 4x(1-x)(q^{\mu}q^{\nu} - g^{\mu\nu}q^{2}) \sum_{n} \frac{c_{n}}{[-k^{2} - q^{2}x(1-x) + m_{n}^{2}]^{2}} \right\} \\
= -\frac{1}{16\pi^{2}} \operatorname{tr}(T^{a}T^{b}) \int_{0}^{1} dx \int_{0}^{\infty} dt \\
\times \left[\Lambda^{2}g^{\mu\nu} \left(2t + t^{2}\frac{d}{dt} \right) \frac{\tilde{f}(t+s)}{t+s} + 4x(1-x)(q^{\mu}q^{\nu} - g^{\mu\nu}q^{2})t \frac{d}{dt} \frac{\tilde{f}(t+s)}{t+s} \right],$$
(6.5)

where $t = -k^{2}/\Lambda^{2}$ and $s = -a^{2}x(1-x)/\Lambda^{2}$

where $t \equiv -k^2/\Lambda^2$ and $s \equiv -q^2x(1-x)/\Lambda^2$.

Now if $\lim_{t\to\infty} t\widetilde{f}(t) = 0$ as in (4.6), the quadratically divergent gauge noninvariant term disappears after a partial integration of the first term. Therefore

^{*} It is important to shift all the integration momenta in the same way. Otherwise, one obtains a gauge non-invariant (but finite) piece which is proportional to $g^{\mu\nu}$.

we obtain the final result

$$\Pi^{\mu\nu ab}(q) = \frac{1}{4\pi^{2}} \operatorname{tr}(T^{a}T^{b}) (q^{\mu}q^{\nu} - g^{\mu\nu}q^{2}) \int_{0}^{1} dx \, x(1-x) \left[\int_{s}^{\infty} dt \, \frac{f(t)}{t} - \int_{s}^{\infty} dt \, \frac{f(t) - \widetilde{f}(t)}{t} \right]
\stackrel{\Lambda \to \infty}{=} \frac{1}{24\pi^{2}} \operatorname{tr}(T^{a}T^{b}) (q^{\mu}q^{\nu} - g^{\mu\nu}q^{2}) \left[\log \frac{\Lambda^{2}}{-q^{2}} + \frac{5}{3} - 2\log \frac{\pi}{2} + \int_{0}^{\infty} dt \, \frac{\widetilde{f}(t) - f(t)}{t} \right],$$
(6.6)

where f(t) is an arbitrary function satisfying (4.6) in the first line, and we have used (4.11) in the last line. Since $f(t) - \widetilde{f}(t) = O(t)$, we have set the limit of the last integral to zero for $\Lambda \to \infty$. This is a general formula for arbitrary $\widetilde{f}(t)$ in (6.2) satisfying (4.6) and reproduces the results in Refs. [1,4]. From the last expression, it is obvious that the coefficient of $\log \Lambda^2$, which gives the fermion contribution to the one loop beta function, is independent of the regulator function (6.2) [4,2]. The last constant, on the other hand, depends on the specific choice of the function (for example, for $\widetilde{f}(t) = f(t)$ in (4.5), the last integral in (6.6) is $2 \log(\pi/2) - \log 2$).

Let us next consider the case of anomaly-free complex representations. In the regularized gauge current (5.5), the second term does not contribute to the vacuum polarization tensor due to (5.9) (n=2). The same calculation as above therefore gives (6.6) with the coefficient $\operatorname{tr}(\mathcal{T}^a\mathcal{T}^b)/(48\pi^2) = \operatorname{tr}(\mathcal{T}^a\mathcal{T}^b)/(24\pi^2)$, the correct result for a chiral fermion in a complex representation T^a .

As expected, the formulas (4.10) and (5.5) give a transverse form without any gauge non-invariant counter terms. Similar calculations of the fermion one loop

[†] If instead, $c_n = (-1)^n$ and $m_n = \sqrt{|n|}\Lambda$ are used in (6.2), $\tilde{f}(t) = t[\Psi((t+1)/2) - \Psi(t/2)] - 1$, where $\Psi(z)$ is the digamma function, and $\lim_{t\to\infty} t\tilde{f}(t) = 1/2 \neq 0$. The partial integration in (6.5) then has an additional surface term, $-\operatorname{tr}(T^aT^b)\Lambda^2g^{\mu\nu}/(32\pi^2)$ (the last integral in (6.6) is $\log(\pi/2)$ [4]). The appearance of the gauge non-invariant piece is somewhat puzzling, because we started with a manifestly gauge invariant expression (4.10). The resolution in the present formulation seems to be the following: When we compute the divergence of the gauge current (4.12), we encounter an integral $\int_0^\infty dt \, t \, \tilde{f}(t)$ (see (7.5)) which is ill-defined for this regulator function. Therefore the WT identity (4.12), and as a result, the transverse condition of the vacuum polarization tensor, cannot be derived with this choice.

vertex functions and a verification of the WT identities [4] may be pursued. Since the regularized form (4.10) and (5.5) for anomaly-free representations are finite and manifestly gauge invariant, the requirements implicit in gauge invariance should automatically be fulfilled.

7. Covariant anomalies

The generalized PV regularization also provides a reliable way to evaluate non-gauge anomalies in the anomaly-free chiral gauge theories. Let us start with the fermion number anomaly [10]: The Majorana-type PV mass term in the formulation naturally provides the source of the fermion number anomaly. We first take directly the divergence of the regularized U(1) current for the real representation (4.17):

$$\partial_{\mu} \langle J^{\mu}(x) \rangle_{PV} = \partial_{\mu} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \not D^{\dagger} \frac{1}{i \not D \not D^{\dagger}} f(\not D \not D^{\dagger} / \Lambda^{2}) \delta(x - y) \right]. \tag{7.1}$$

To evaluate this, we introduce the normalized eigenfunctions of the hermitian operators, $\not \!\! D^{\dagger} \not \!\! D$ and $\not \!\! D \not \!\!\! D^{\dagger}$ [12],

$$\mathcal{D}^{\dagger}\mathcal{D}\varphi_n(x) = \lambda_n^2 \varphi_n(x), \quad \mathcal{D}\mathcal{D}^{\dagger}\varphi_n(x) = \lambda_n^2 \varphi_n(x), \tag{7.2}$$

where λ_n is real and positive. From this definition (by appropriately choosing the phase) we see

We first use the completeness relation of $\phi_n(x)$, $\sum_n \phi_n(x) \phi_n^{\dagger}(y) = \delta(x-y)$, in (7.1).

[‡] If we had $f(\not D^2/\Lambda^2)$ instead in (4.17) (this regulator is known [16] to give the consistent anomaly), we would obtain no fermion number anomaly, $\partial_{\mu} \langle J^{\mu}(x) \rangle = 0$.

The calculation then proceeds as follows:

where we have used (7.3) in several steps. The subsequent calculation is identical to the anomaly evaluation in the path integral framework [12]: i) $\delta(x-y) = \int d^4k \, e^{ik(x-y)}/(2\pi)^4$, ii) shift e^{ikx} to the left, iii) scale $k_{\mu} \to \Lambda k_{\mu}$ and expand $f(\mathcal{D}\mathcal{D}^{\dagger}/\Lambda^2)$ by $1/\Lambda$. Finally by using (4.6), we have

$$\operatorname{tr} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \left\{ f(\not{\!\!D} \not{\!\!D}^{\dagger}/\Lambda^2) \right\} e^{ikx}$$

$$\stackrel{\Lambda \to \infty}{=} \frac{1}{4\pi^2} \int_0^{\infty} dt \, t f(t) \dim T\Lambda^4 \mp \frac{g^2}{64\pi^2} \varepsilon^{\mu\nu\rho\sigma} \operatorname{tr} \left(F_{\mu\nu} F_{\rho\sigma} \right) + \frac{g^2}{48\pi^2} \operatorname{tr} \left(F_{\mu\nu} F^{\mu\nu} \right), \tag{7.5}$$

and consequently,

$$\partial_{\mu} \langle J^{\mu}(x) \rangle_{PV} = -i \operatorname{tr} \int \frac{d^{4}k}{(2\pi)^{4}} e^{-ikx} [f(\not D \not D^{\dagger}/\Lambda^{2}) - f(\not D^{\dagger} \not D/\Lambda^{2})] e^{ikx}$$

$$\stackrel{\Lambda \to \infty}{=} \frac{ig^{2}}{32\pi^{2}} \varepsilon^{\mu\nu\rho\sigma} \operatorname{tr} (F_{\mu\nu} F_{\rho\sigma}).$$
(7.6)

This reproduces the well-known gauge invariant form of the fermion number anomaly [10].

[§] Since the fermion number U(1) rotation and the non-Abelian gauge transformation commute, the Wess-Zumino consistency condition [7] implies a gauge invariance of the fermion number anomaly.

We may also compute the fermion number anomaly using the WT identity (3.9). It can be verified that direct evaluation of the right-hand side of the equation again gives the correct anomaly. Namely,

$$\langle B(x) \rangle_{PV} = i \lim_{y \to x} \operatorname{tr} \left(\frac{M^{\dagger} M}{\not \!\!\!D \not \!\!\!D^{\dagger} + M^{\dagger} M} - \frac{M'^{\dagger} M'}{\not \!\!\!D \not \!\!\!D^{\dagger} + M'^{\dagger} M'} - \frac{M'^{\dagger} M'}{\not \!\!\!D^{\dagger} \not \!\!\!D^{\dagger} + M'^{\dagger} M'} + \frac{M'^{\dagger} M'}{\not \!\!\!D^{\dagger} \not \!\!\!D + M'^{\dagger} M'} \right) \delta(x - y).$$

$$(7.7)$$

It is obvious that this leads to the last line of (7.4), and thus (7.6).

For the anomaly-free complex representation, the regularized U(1) current is given by (5.6). However, by repeating the above manipulations, it is easy to see that the last, seemingly unregularized term does not contribute to the anomaly. Namely, the divergence of the last term identically vanishes. The divergence of the first term of (5.6), on the other hand, gives one half of (7.6), but with the doubled gauge generator (5.1) (note the structure constant is common for T^a and $-T^{a*}$). Thus the fermion number anomaly in the complex representation also results in (7.6), when rewritten in terms of the original gauge generator T^a .

Let us next consider the conformal anomaly represented by (4.18) and (5.7). From (7.5), for both cases we have

$$\langle T^{\mu}_{\mu}(x)\rangle_{PV} = \frac{1}{2}\operatorname{tr} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} [f(\not \mathbb{D} \not \mathbb{D}^{\dagger}/\Lambda^2) + f(\not \mathbb{D}^{\dagger} \not \mathbb{D}/\Lambda^2)] e^{ikx}$$

$$\stackrel{\Lambda \to \infty}{=} \frac{1}{4\pi^2} \int_{0}^{\infty} dt \, t f(t) \dim T\Lambda^4 + \frac{g^2}{48\pi^2} \operatorname{tr} (F_{\mu\nu} F^{\mu\nu}), \tag{7.8}$$

which reproduces the correct gauge invariant result [11,13].

8. Relation to the vector-like formalism

Finally we briefly comment on the relation of the present Lagrangian (2.1), which may be called the Weyl formulation, to the generalized PV regularization proposed by Narayanan and Neuberger [3], the so-called vector-like formulation.

Let us begin with the complex representations. We introduce new variables [4]

$$\chi_R \equiv \frac{1+\sigma^3}{2} P_R \begin{pmatrix} \psi_0 \\ \psi_2 \\ \vdots \end{pmatrix}, \quad \chi_L \equiv i\sigma^2 \frac{1-\sigma^3}{2} P_L C_D \begin{pmatrix} \overline{\psi}_2^T \\ \overline{\psi}_4^T \\ \vdots \end{pmatrix}, \tag{8.1}$$

and

$$\varphi_R \equiv \frac{1 + \sigma^3}{2} P_R \begin{pmatrix} \phi_1 \\ \phi_3 \\ \vdots \end{pmatrix}, \quad \varphi_L \equiv i\sigma^2 \frac{1 - \sigma^3}{2} P_L C_D \begin{pmatrix} \overline{\psi}_1^T \\ \overline{\psi}_3^T \\ \vdots \end{pmatrix}, \tag{8.2}$$

where we have assigned even generation indices for the fermions, and odd number indices for bosons. Then the PV Lagrangian (2.1) with (5.2) and (5.4), discarding the left handed spectator fields, is rewritten in a vector-like form of Ref. [3]

$$\mathcal{L} = \overline{\chi} D \chi - \overline{\chi} (N P_R + N^{\dagger} P_L) \chi + \overline{\varphi} D \varphi - \overline{\varphi} N' \varphi, \tag{8.3}$$

where the covariant derivative is vector-like:

$$D \equiv \gamma^{\mu} (\partial_{\mu} - igA_{\mu}^{a}T^{a}). \tag{8.4}$$

Reflecting the fact that the original theory is chiral, the mass matrix N has a non-trivial analytic index [3], dim ker $N^{\dagger}N$ – dim ker $NN^{\dagger}=1$:

$$N \equiv \begin{pmatrix} 0 & 2 & & & \\ & 0 & 4 & & \\ & & 0 & 6 & \\ & & & \ddots & \ddots \end{pmatrix} \Lambda, \quad N' \equiv \begin{pmatrix} 1 & & & \\ & 3 & & \\ & & 5 & \\ & & & \ddots \end{pmatrix} \Lambda. \tag{8.5}$$

Therefore, we see that the generalization [5] of the Weyl formulation to arbitrary complex representations is basically equivalent to the vector-like formulation.

How does the anomaly-free requirement emerge in the vector-like formulation? According to Fujikawa [2], the regularized gauge current for (8.3) is expressed as

$$\langle J^{\mu a}(x)\rangle = \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} T^{a} \frac{1}{i \not \!\!\!D} f(\not \!\!\!D^{2}/\Lambda^{2}) \delta(x-y) \right] + \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \gamma_{5} T^{a} \frac{1}{i \not \!\!\!D} \delta(x-y) \right]. \tag{8.6}$$

Similarly, the axial U(1) current associated with $\chi(x) \to e^{i\alpha\gamma_5}\chi(x)$ and $\varphi(x) \to e^{i\alpha\gamma_5}\varphi(x)$ is

$$\left\langle J_5^{\mu}(x)\right\rangle = \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \gamma_5 \frac{1}{i \not \!\!\!D} f(\not \!\!\!D^2/\Lambda^2) \delta(x-y) \right] + \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \frac{1}{i \not \!\!\!D} \delta(x-y) \right], \tag{8.7}$$

and the vector U(1) current associated with $\chi(x) \to e^{i\alpha}\chi(x)$ and $\varphi(x) \to e^{i\alpha}\varphi(x)$ is,

$$\left\langle \widetilde{J}^{\mu}(x) \right\rangle = \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \frac{1}{i \not \!\!\!D} f(\not \!\!\!D^2/\Lambda^2) \delta(x-y) \right] + \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \gamma_5 \frac{1}{i \not \!\!\!D} \delta(x-y) \right]. \tag{8.8}$$

For non-anomalous gauge representations, it can be argued that the first two currents (8.6) and (8.7) are in fact regularized [3,4,2]. Demonstration of this point requires a somewhat detailed form of the Feynman integral and goes as follows: The last terms in (8.6)–(8.8) are evaluated as

$$\lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \begin{Bmatrix} T^{a} \gamma_{5} \\ 1 \\ \gamma_{5} \end{Bmatrix} \frac{1}{i \cancel{p}} \delta(x - y) \right]$$

$$= -\sum_{n=0}^{\infty} g^{n} \int \frac{d^{4}k_{1}}{(2\pi)^{4}} \cdots \frac{d^{4}k_{n}}{(2\pi)^{4}} \int dx_{1} A_{\mu_{1}}^{a_{1}}(x_{1}) e^{ik_{1}(x - x_{1})} \cdots \int dx_{n} A_{\mu_{n}}^{a_{n}}(x_{n}) e^{ik_{n}(x - x_{n})}$$

$$\times \operatorname{tr} \left(\begin{Bmatrix} T^{a} \\ 1 \\ 1 \end{Bmatrix} T^{a_{1}} \cdots T^{a_{n}} \right)$$

$$\times \int \frac{d^{4}k}{(2\pi)^{4}} \operatorname{tr} \left(\gamma^{\mu} \begin{Bmatrix} \gamma_{5} \\ 1 \\ \gamma_{5} \end{Bmatrix} \frac{1}{\cancel{k} + \cancel{k}_{1} + \cdots + \cancel{k}_{n}} \gamma^{\mu_{1}} \frac{1}{\cancel{k} + \cancel{k}_{2} + \cdots + \cancel{k}_{n}} \gamma^{\mu_{2}} \cdots \gamma^{\mu_{n}} \frac{1}{\cancel{k}} \right). \tag{8.9}$$

We then change the integration variable k_{μ} to

$$k_{\mu} \to -k_{\mu} - k_{1\mu} - k_{2\mu} - \dots - k_{n\mu}$$
 (8.10)

and insert $C_D^{-1}C_D=1$ in the trace. Shifting C_D to the right-hand side transposes the gamma matrices, and it can be expressed as a transpose of the product of gamma matrices. Finally, by renaming all the subscripts $(1, 2, \dots, n) \to (n, \dots, 2, 1)$, we see the integrand in (8.9) is proportional to

$$\operatorname{tr}\left(\left\{\begin{array}{c} T^{a} \\ 1 \\ 1 \end{array}\right\} T^{a_{1}} \cdots T^{a_{n}}\right) + (-1)^{n} \operatorname{tr}\left(\left\{\begin{array}{c} T^{a} \\ -1 \\ 1 \end{array}\right\} T^{a_{n}} \cdots T^{a_{1}}\right)$$

$$\times \int \frac{d^{4}k}{(2\pi)^{4}} \operatorname{tr}\left(\gamma^{\mu} \left\{\begin{array}{c} \gamma_{5} \\ 1 \\ \gamma_{5} \end{array}\right\} \frac{1}{\not{k} + \not{k}_{1} + \dots + \not{k}_{n}} \gamma^{\mu_{1}} \frac{1}{\not{k} + \not{k}_{2} + \dots + \not{k}_{n}} \gamma^{\mu_{2}} \cdots \gamma^{\mu_{n}} \frac{1}{\not{k}}\right). \tag{8.11}$$

For $n \leq 3$ the momentum integration is power counting divergent, but in the *first* two lines, the coefficient for $n \leq 3$ vanishes provided that (5.9), which is equivalent to $\operatorname{tr} T^a = \operatorname{tr}(T^a\{T^b, T^c\}) = 0$, holds. Therefore for anomaly-free representations, it can be argued that the last terms in (8.6) and (8.7) are finite. On the other hand, the coefficient of the last line of (8.11) is proportional to $\operatorname{tr}(T^aT^b) \neq 0$ for n=2 and n=3, and the above argument cannot be applied (J_5^{μ}) are different objects).

What corresponds to the unregularized U(1) current $\widetilde{J}^{\mu}(x)$ in the Weyl formulation (2.1)? It is the Noether current associated with a U(1) rotation, $\psi(x) \to e^{i\alpha\sigma^3}\psi(x)$ and $\phi(x) \to e^{i\alpha\sigma^3}\phi(x)$. Since the total Lagrangian is invariant under this rotation, the current is conserved $(\partial_{\mu}\widetilde{J}^{\mu}(x) = 0)$ and thus cannot be used as the fermion number current (which should be anomalous). In fact in the Weyl formulation,

$$\begin{split} & \left\langle \widetilde{J}^{\mu}(x) \right\rangle_{PV} \equiv \left\langle \overline{\psi} \gamma^{\mu} \sigma^{3} \psi(x) + \overline{\phi} \gamma^{\mu} \sigma^{3} \phi(x) \right\rangle \\ &= \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \sigma^{3} \frac{1}{i \not \!\!\!D} f(\not \!\!\!D \not \!\!\!D^{\dagger} / \Lambda^{2}) \delta(x-y) \right] + \frac{1}{2} \lim_{y \to x} \operatorname{tr} \left[\gamma^{\mu} \frac{1}{i \not \!\!\!D} \delta(x-y) \right], \end{split} \tag{8.12}$$

and the last term is not regularized (note \mathcal{D} in this expression contains $P_R = (1 + \gamma_5)/2$).

We have observed that the Weyl formulation in §5 (complex representation) is basically equivalent to the vector-like formulation in Ref. [3]. An advantage of the Weyl formulation is, however, that the requirement of the anomaly-free nature emerges in a rather simple way; it requires only a power-counting, and the above argument based on the change of the momentum integration variable and the charge conjugation invariance is effectively shortcut by the introduction of \mathcal{T}^a , the doubled representation, and the matrix σ^3 .

Let us now turn to real representations. It is obvious that for real-positive representations, the Weyl formulation can be non-equivalent to the vector-like formulation because it only requires a finite number of regulator fields ((8.3)) on the other hand always requires an infinite number of such fields to have a non-trivial analytic index). Also, for the real representations we can construct the Lagrangian such that the seemingly unregularized terms do not appear from the beginning ((4.10) and (4.17)). In the Weyl formulation, the fact that all the real representations have no gauge anomaly can be incorporated into the Lagrangian construction. This is another advantage of the Weyl formulation.

9. Conclusion

In this paper, we have studied the general structure of the generalized PV regularization proposed by Frolov and Slavnov on the basis of a regularization of composite operators. We have observed that the PV Lagrangian provides a gauge-invariant regularization of the chiral fermion in arbitrary anomaly-free gauge representations. The generalization [5] to the arbitrary complex representation is basically equivalent to the vector-like formulation in Ref. [3], and real representations can be treated in a straightforward manner. As the gauge current is regularized in a gauge invariant way, the vacuum polarization tensor, for example, is found to be transverse. We have also computed the fermion number anomaly and the

conformal anomaly within our formulation and the gauge invariant form of the anomalies were reproduced.

A practical calculation of multi-point vertex functions is simpler if one starts directly from the covariant regularization [9], because one can then choose a convenient form of the regulator function f(t). Nevertheless, the very existence of a Lagrangian level gauge invariant regularization makes the renormalizability and the unitarity proofs of the anomaly-free chiral gauge theories (at least conceptually) simpler: No gauge non-invariant counter term is needed to compensate for the breaking of gauge symmetry by the regularization. In particular, the standard model may be directly treated in the scheme.

It seems to us, however, that the real importance of a possibility to construct such a Lagrangian level gauge invariant regularization lies in a possible implication on the lattice chiral gauge theory, in which a consistent treatment of chiral fermions has been a long standing problem. In fact, several proposal have been made on the basis of the generalized PV Lagrangian [17]. A remark on this problem is found in Ref. [18].

It would be interesting to consider a supersymmetric extension of the generalized PV regularization, as a supersymmetric, gauge invariant one-loop regularization.

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